Time Varying Fields

Dr. M.A.Motawea

From our previous studies, it is clear that:

$\nabla X E = 0$

$\nabla X H = J$

$\nabla . D = \rho$

$\nabla . B = o$

Maxwell's equations in static case

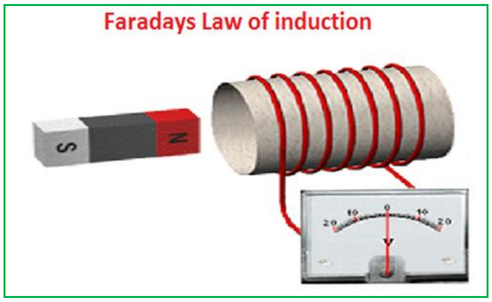
A new concept will be introduced:

- The electric field strength, **E** produced by changing magnetic field strength, **H**. Faraday's law
- The magnetic field strength, **H** produced by changing electric field strength, **E** . Amper's law

This is done experimentally by **Faraday** and theoretical efforts of **Maxwell**.

- **Faraday's Law**: (by experimental work)

If a conductor moves in a magnetic field or the magnetic field changes, there will be electromotive force , e.m.f. or voltage arises in the conductor.



Faraday's law stated as:

$$e.m.f = -\frac{d}{dt}\phi \quad [V] \qquad (1)$$

This equation implies a closed path.

e.m.f. in such a direction produces current has flux if added to the original flux , this will reduce the magnitude of e.m.f. (this statement, induced voltage produces opposing flux known as **Lenz's** law).

If the closed path taken by N turns conductor, then,

$$e.m.f = -N\frac{d}{dt}\phi \quad [V] \qquad (2)$$

We define:

$$e.m.f = \oint_C E.dl \quad [V] \qquad (3)$$

Also, magnetic flux

$$\phi = \int B.dS$$

Then,

$$\oint_C E.dl = -\frac{\partial}{\partial t}\int_S B.dS \qquad (4)$$

This equation is the integral form of 1st Maxwell's equation.

Apply Stokes theorem:

$$\oint_C E.dl = \int_S (\nabla X E).dS$$

Eq^n (4) becomes:

$$\int_S (\nabla X E).dS = -\frac{\partial}{\partial t}\int_S B.dS$$

i.e.

$$\nabla X E = -\frac{\partial}{\partial t} B \qquad (5)$$

This is 1st Maxwell's equation in differential or point form.

Which means that a time- changing magnetic field **B(t**) produces an electric field, **E**, it has a property of circulation, its line integral about a general closed path isn't zero.

If B is not function of time (i.e. static form), then:

$$\oint_C E.dl = 0 \quad \textbf{and} \quad \nabla X E = 0$$

As denoted above.

- **Continuity Equation:**

If we consider a region bounded by a closed surface s, the current through the closed surface is : $I = \oint_S J.dS$

If the charge inside a closed surface is denoted by Q_i, then the rate of decreasing is

$$-\frac{d}{dt}Q_i \quad ,$$

and the principle of conservation of charge requires :

$$I = \oint_S J.dS = -\frac{d}{dt}Q_i = -\frac{d}{dt}\int_V \rho_V dV \qquad (6)$$

Equation (6) is the continuity equation in integral form.

By using divergence theorem :

$$\oint_S J.dS = \int_V (\nabla.J).dV$$

$$\int_V (\nabla.J).dV = -\frac{d}{dt}\int_V \rho_V dV \qquad (7)$$

$$\therefore (\nabla . J) = -\frac{\partial}{\partial t}\rho_V \qquad (8)$$

Equation (8) is the continuity equation of current in point form.

- **Displacement Current :**

- **conduction current** occurs in the presence of electric field **E** within **a conductor** of fixed cross section with conductivity σ where :

$$J_C = \sigma E \qquad (9)$$

- also **displacement current** occurs within a **dielectric material**,

$$J_d = \frac{\partial}{\partial t} D \qquad (10)$$

- some materials have both currents, J_C and J_d

- You know at steady state, for magnetic field **H**, **Ampere's circuital law is:**

$$\nabla X H = J_C \qquad (11)$$

Taking divergence, then:

$$\nabla . (\nabla X H) = \nabla . J_C \qquad (12)$$

$$0 = \nabla . J_C$$

But we have eq^n (8):

$$\nabla . J = -\frac{\partial}{\partial t}\rho_V$$

so we must add term **G** to eq^n. 11, then

$$\nabla X H = J_C + G \qquad (13)$$

By taking $\nabla.$

So, $$\nabla.(\nabla X H) = \nabla.J_C + \nabla.G$$

Then, $$0 = \nabla.J_C + \nabla.G$$

$$\nabla.G = -\nabla.J = \frac{\partial}{\partial t}\rho \qquad (14)$$

but we have **Gauss law:**

$$\nabla.D = \rho$$

$$\therefore \nabla.G = \frac{\partial}{\partial t}\nabla.D \qquad \text{i.e} \qquad \nabla.G = \nabla.\frac{\partial}{\partial t}D$$

then $$G = \frac{\partial}{\partial t}D \qquad (15)$$

Eq ^{n} (13) becomes:

$$\nabla X H = J_C + \frac{\partial}{\partial t}D \qquad (16)$$

That's the second Maxwell's equation (Amper's circuital law in point form).

Apply Stokes theorem:

$$\oint_C H.dl = \int_S (\nabla X H).dS$$

Eq ^{n} (16) becomes:

$$\oint_C H.dl = \int_S (J_C + J_D).dS \qquad (17)$$

That's the second Maxwell's equation (Amper's circuital law in integral form).

- **Maxwell's Equation in point form:**

$$\nabla XE = -\frac{\partial}{\partial t} B$$ Faraday's Law

$$\nabla XH = J_C + \frac{\partial}{\partial t} D$$ Amper's Law

$$\nabla .D = \rho_V$$ Gaussian Law, for electric

$$\nabla .B = 0$$ Gaussian Law, for electric

These four equations are the basic of electromagnetic theory .

They are partial differential equations related E & H to each other, to their sources (charges and current density).

There are some auxiliary equations:

$$J_C = \sigma E\,, \qquad D = \varepsilon E\,, \qquad B = \mu H$$

- **Maxwell's Equations in integral form:**

we have:

1.

$$\nabla XE = -\frac{\partial}{\partial t} B$$ Faraday's Law in pt. form

By using Stokes theorem:

$$\oint_C E.dl = \int_S (\nabla X E).dS$$

$$\oint_C E.dl = -\frac{\partial}{\partial t}\int_S B.dS$$ Faraday's Law in intg. form

2.

$$\nabla X H = J_C + \frac{\partial}{\partial t} D$$ Amper's Law in pt. form

By using Stokes theorem:

$$\oint_C H.dl = \int_S (\nabla X H).dS$$

$$\oint_C H.dl = \int_S (J_C + \frac{\partial}{\partial t} D).dS$$ Amper's Law in intg. Form

3,4- **By using Divergence theorem,**

- **Gauss law for electricity is:**

$$\oint_S D.dS = \int_V \rho_V dV = Q_{en}$$ Gauss law in integral form

- **Gauss law for magnetic is:**

$$\oint_S B.dS = 0$$ Gauss law in integral form

These equations are used to determine B.C. (tgt & normal components of fields, E,D,H, and B) between two media.

Meaning of Maxwell's equations:

1- **The first law states that e.m.f around a closed path is equal to the inflow of magnetic current through any surface bounded by the path.**

2- **The second law states that magneto motive force m.m.f. around a closed path is equal to the sum of electric displacement and, conduction currents through any surface bounded by the path.**

3- **The third law states that the total electric displacement flux passing through a closed surface (Gaussian surface) is equal to the total charge inside the surface.**

4- **The fourth law states that the total magnetic flux passing through any closed surface is zero.**

H = magnetic field strength, [A/m]

D = electric flux density, [C/m]

$\frac{\partial}{\partial t}D$ = displacement current density, [A/m^2]

J = conduction current density, [A/m^2]

E = electric field [V/m]

B = magnetic flux density, [wb/m^2] or Tesla

$\frac{\partial}{\partial t}B$ = time derivative of magnetic flux density, [wb/(m^2 .sec)], or Tesla/sec

Boundary conditions are :

$\mathbf{E}_{t1}=\mathbf{E}_{t2}$: tgt component of electric field strength is continuous on the interface between 2 media.

$\mathbf{D}_{n1}-\mathbf{D}_{n2}=\boldsymbol{\rho}$: normal component of electric flux density equal the charge on the surface between 2 media.

H_{t1}= H_{t2}: tgt component of magnetic field strength is continuous on the interface between 2 media.

B_{n1}= B_{n2} normal component of magnetic flux density is continuous on the interface between 2 media.

properties of the medium:

- Properties of the medium is characterized by parameters such as ε, μ, σ
- Medium is classified into: linear, isotropic, and homogeneous medium where:
  1. **Linear** medium has ε, μ, σ are not f^n of E &H
  2. **Isotropic** medium has **J** parallel to **E** & **B** parallel to **H** and **D** parallel to **B**
  3. **Homogeneous** medium has ε, μ, σ are constant and not f^n of coodt.
  4. **Free space** has neither charges nor current , it has ε_0, μ_0

Hint:

- From gauss's law in electric field, we have:

$$\oint_S D.dS = Q = \int_V \rho_V dV$$

apply divergence theorem, $\int_V (\nabla.D).dV = \int_V \rho_V dV$

we get:++++++++++++++++++++

$$\therefore \nabla.D = \rho_V$$

- From gauss's law in magnetic field, we have:

$$\oint_S B.dS = 0$$

apply divergence theorem, $\int_V (\nabla.B).dV = 0$

$$\therefore \nabla.B = 0$$

Maxwell's equations

- ***Amper's Law:***

$$\oint_C H.dl = \int_S (J_C + \frac{\partial}{\partial t} D).dS$$ integral form

By using Stokes theorem:

$$\oint_C H.dl = \int_S (\nabla X H).dS$$ then, $$\nabla X H = J_C + \frac{\partial}{\partial t} D$$ point form

- ***Faraday's Law:***

$$\oint_C E.dl = -\frac{\partial}{\partial t} \int_S B.dS$$ integral form

apply Stokes theorem:

$$\oint_C E.dl = \int_S (\nabla X E).dS$$ Then, $$\nabla X E = -\frac{\partial}{\partial t} B$$ point form

- ***Gauss Law:***

a- **For electric field** $$\oint_S D.dS = Q = \int_V \rho_V dV$$ integral form

apply divergence theorem: $$\int_V (\nabla.D).dV = \int_V \rho_V dV$$ then, $$\nabla.D = \rho_V$$ point form

b- **For magnetic field** $$\oint_S B.dS = 0$$ integral form

apply divergence theorem: $$\int_V (\nabla.B).dV = 0$$ then, $$\nabla.B = 0$$ point form

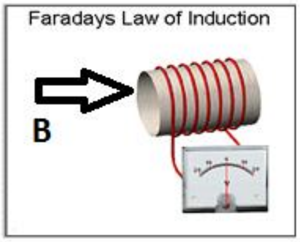
Microwave Engineering

Sheet #1

(4/10/2015)

Time Varying Fields

Q1: A circular loop of 10 cm radius is located in the xy plane in B field given by: $\boldsymbol{B} = (0.5 \cos 377\, t)(3\, \boldsymbol{a}_y + 4\, \boldsymbol{a}_z)\ T$. **Determine**: the voltage induced in the loop ?



Q2: **Find** the displacement current density for :

- Next to your radio where the local AM station provides a field strength of $\boldsymbol{E} = 0.02 \sin [0.1927\, (3\text{x}10^8 t - z)]\ \boldsymbol{a}_x$
- In a good conductor where σ 10^7S/m and the conduction current density is high, like $10^7 \sin (120\, \pi t)\ \boldsymbol{a}_x\ A/m^2$.

Q3: An inductor is formed by winding 10 N turns of a thin wire around a wooden rod which has a radius of 2 cm, If a uniform , sinusoidal magnetic field with magnitude 0.01 wb/m^2 and frequency of 10 KHz is directed along the axis of the rod . **Determine:** the voltage induced between the two ends of the wire assuming the two ends are closed together?

Q4 : A material having a conductivity σ and permittivity ε is placed in a sinusoidal, time-varying electric field having a frequency ω. **At what** frequency will the conduction current equal to the displacement current? If, $\sigma = 10^{-12}$S/m ,and $\varepsilon = 3\varepsilon_0$.

Q5 :Show that the fields: $\boldsymbol{E} = E_m \sin X \sin t\ \boldsymbol{a}_y$, $\boldsymbol{H} = (E_m/\mu) \cos X \cos t\ \boldsymbol{a}_z$ in free space satisfy Faraday's law and the two laws of Gauss but don't satisfy Ampere's law.

Q6: If $\boldsymbol{E}$ of radio broadcast signal at T.V Rx is given by: $\boldsymbol{E} = 5 \cos(\omega t - \beta y) a_z$

Determine: the displacement current density.

If the same field exists in a medium whose conductivity σ is given by : σ =

$2*10^3[\Omega^{-1}/cm]$, **<u>Find:</u>** the conduction current density?

Q8: Given ***E***= *10 sin* $(\omega t - \beta z)a_y$ [v/m] in free space, **Find**: **D,B** and **H**

Q9: a parallel plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage 50 sin 10^3t [v] applied to its plates. **Calculate:** the displacement current assuming $\varepsilon = 2\varepsilon_o$

Q10: Show that the following fields vector in free space satisfy all Maxwell's equations, $E = E_0 \cos(\omega t - \beta z)a_x$, $H = \frac{E_0}{\eta}\cos(\omega t - \beta z)a_y$

Q11: A perfectly conducting sphere of radius R in free space has a charge Q uniformly distributed over its surface , utilizing B.C. **Determine** the electric field E at the surface of the sphere , **show that** by using Gauss law, the result is correct.

Good luck, *Dr. M.A.Motawea*

Solution

Q1: $\boldsymbol{B} = (0.5 \cos 377\, t)(3\, \boldsymbol{a}_y + 4\, \boldsymbol{a}_z)\ T$

$$emf = -N\frac{d\Phi}{dt} \quad [V] \quad = -NA\frac{dB}{dt} = \frac{377}{2}10^{-2}\pi \sin 377t(3a_y + 4a_z)Tesla$$

$$= 1.88 \sin 377t(3a_y + 4a_z)T \qquad , N=1$$

$$|e.m.f| = 1.88V$$

Q2:

- $\boldsymbol{E} = 0.02 \sin [0.1927 (3x10^8 t - z)]\ \boldsymbol{a}_x$

$$J_d = \frac{\partial}{\partial t}D = \varepsilon_0 \frac{\partial}{\partial t}E = \frac{10^{-9}}{36\pi}0.02 * 0.1927 * 3 * 10^8 \cos[0.1927(3*10^8 t - z)]a_x$$

$$= 1.008 * 10^{-4} \cos[--------]a_x$$

$$|J_d| = 0.1mA$$

- For good conductor

Q8:

E = 10 sin (ωt - βy) a_y, V/m
D = $\in_0$ E, $\in_0$ = 8.854 x 10-12 F/m
D = 10$\in_0$ sin (ωt - βy) a_y, C/m2

Second Maxwell's equation is: ∇ x E = -B

As E_y = 10 sin (ωt - βz) V/m

Now, ∇ x E becomes

= 10 β cos (ωt - βz) a_x

That is, $\nabla X E = \begin{vmatrix} a_x a_y a_z \\ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \\ 0 E_y 0 \\ \end{vmatrix}$

Or