Automatic Control Systems (FCS)

Lecture- 8
Steady State Error
Introduction

• Any physical control system inherently suffers steady-state error in response to certain types of inputs.

• A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input.

• Whether a given system will exhibit steady-state error for a given type of input depends on the type of open-loop transfer function of the system.
Classification of Control Systems

• Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs, and so on.

• The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.
Classification of Control Systems

• Consider the unity-feedback control system with the following open-loop transfer function:

\[ G(s) = \frac{K(T_1s + 1)(T_2s + 1)\ldots(T_ms + 1)}{s^N(T_1s + 1)(T_2s + 1)\ldots(T_ps + 1)} \]

• It involves the term \( s^N \) in the denominator, representing \( N \) poles at the origin.

• A system is called type 0, type 1, type 2, ... , if \( N=0, N=1, N=2, ... \) , respectively.
Classification of Control Systems

• As the type number is increased, accuracy is improved.

• However, increasing the type number aggravates the stability problem.

• A compromise between steady-state accuracy and relative stability is always necessary.
Steady State Error of Unity Feedback Systems

• Consider the system shown in following figure.

• The closed-loop transfer function is

\[
\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}
\]
Steady State Error of Unity Feedback Systems

• The transfer function between the error signal $E(s)$ and the input signal $R(s)$ is

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)}$$

• The final-value theorem provides a convenient way to find the steady-state performance of a stable system.

• Since $E(s)$ is

$$E(s) = \frac{1}{1 + G(s)} R(s)$$

• The steady state error is

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$
Static Error Constants

• The static error constants are figures of merit of control systems. The higher the constants, the smaller the steady-state error.

• In a given system, the output may be the position, velocity, pressure, temperature, or the like.

• Therefore, in what follows, we shall call the output “position,” the rate of change of the output “velocity,” and so on.

• This means that in a temperature control system “position” represents the output temperature, “velocity” represents the rate of change of the output temperature, and so on.
**Static Position Error Constant (K_p)**

- The steady-state error of the system for a unit-step input is

\[ e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s} = \frac{1}{1 + G(0)} \]

- The static position error constant \( K_p \) is defined by

\[ K_p = \lim_{s \to 0} G(s) = G(0) \]

- Thus, the steady-state error in terms of the static position error constant \( K_p \) is given by

\[ e_{ss} = \frac{1}{1 + K_p} \]
Static Position Error Constant ($K_p$)

- For a **Type 0** system

  $$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1) \cdots}{(T_1 s + 1)(T_2 s + 1) \cdots} = K$$

- For **Type 1** or higher systems

  $$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1) \cdots}{s^N(T_1 s + 1)(T_2 s + 1) \cdots} = \infty, \quad \text{for } N \geq 1$$

- For a unit step input the steady state error $e_{ss}$ is

  $$e_{ss} = \frac{1}{1 + K}, \quad \text{for type 0 systems}$$

  $$e_{ss} = 0, \quad \text{for type 1 or higher systems}$$
The steady-state error of the system for a unit-ramp input is

\[ e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^2} \]

\[ = \lim_{s \to 0} \frac{1}{sG(s)} \]

The static position error constant \( K_v \) is defined by

\[ K_v = \lim_{s \to 0} sG(s) \]

Thus, the steady-state error in terms of the static velocity error constant \( K_v \) is given by

\[ e_{ss} = \frac{1}{K_v} \]
Static Velocity Error Constant ($K_v$)

- For a **Type 0** system

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1) \cdots}{(T_1 s + 1)(T_2 s + 1) \cdots} = 0$$

- For **Type 1** systems

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1) \cdots}{s(T_1 s + 1)(T_2 s + 1) \cdots} = K$$

- For type 2 or higher systems

$$K_v = \lim_{s \to 0} \frac{sK(T_a s + 1)(T_b s + 1) \cdots}{s^N(T_1 s + 1)(T_2 s + 1) \cdots} = \infty, \quad \text{for } N \geq 2$$
Static Velocity Error Constant \( (K_v) \)

- For a ramp input the steady state error \( e_{ss} \) is

\[
e_{ss} = \frac{1}{K_v} = \infty, \quad \text{for type 0 systems}
\]
\[
e_{ss} = \frac{1}{K_v} = \frac{1}{K}, \quad \text{for type 1 systems}
\]
\[
e_{ss} = \frac{1}{K_v} = 0, \quad \text{for type 2 or higher systems}
\]
Static Acceleration Error Constant \((K_a)\)

- The steady-state error of the system for parabolic input is

\[
e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s^3}
\]

\[
= \lim_{s \to 0} \frac{1}{s^2G(s)}
\]

- The static acceleration error constant \(K_a\) is defined by

\[
K_a = \lim_{s \to 0} s^2G(s)
\]

- Thus, the steady-state error in terms of the static acceleration error constant \(K_a\) is given by

\[
e_{ss} = \frac{1}{K_a}
\]
Static Acceleration Error Constant ($K_a$)

- For a **Type 0** system
  \[ K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0 \]

- For **Type 1** systems
  \[ K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = 0 \]

- For **type 2** systems
  \[ K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^2(T_1 s + 1)(T_2 s + 1)\cdots} = K \]

- For **type 3** or higher systems
  \[ K_a = \lim_{s \to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \quad \text{for } N \geq 3 \]
Static Acceleration Error Constant ($K_a$)

- For a parabolic input the steady state error $e_{ss}$ is

$$e_{ss} = \infty, \quad \text{for type 0 and type 1 systems}$$

$$e_{ss} = \frac{1}{K}, \quad \text{for type 2 systems}$$

$$e_{ss} = 0, \quad \text{for type 3 or higher systems}$$
## Summary

<table>
<thead>
<tr>
<th></th>
<th>Step Input ( r(t) = 1 )</th>
<th>Ramp Input ( r(t) = t )</th>
<th>Acceleration Input ( r(t) = \frac{1}{2}t^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 0 system</td>
<td>( \frac{1}{1 + K} )</td>
<td>( \infty )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>Type 1 system</td>
<td>0</td>
<td>( \frac{1}{K} )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>Type 2 system</td>
<td>0</td>
<td>0</td>
<td>( \frac{1}{K} )</td>
</tr>
</tbody>
</table>
Example#1

- For the system shown in figure below evaluate the static error constants and find the expected steady state errors for the standard step, ramp and parabolic inputs.
Example#1 (Steady State Errors)

\[ K_p = \infty \quad K_v = \infty \quad K_a = 10.4 \]

\[ e_{ss} = \frac{1}{1 + K_p} = 0 \]

\[ e_{ss} = \frac{1}{K_v} = 0 \]

\[ e_{ss} = \frac{1}{K_a} = 0.09 \]
Example#1 (evaluation of Static Error Constants)

\[ G(s) = \frac{100(s + 2)(s + 5)}{s^2(s + 8)(s + 12)} \]

\[ K_p = \lim_{{s \to 0}} G(s) \]

\[ K_p = \lim_{{s \to 0}} \left( \frac{100(s + 2)(s + 5)}{s^2(s + 8)(s + 12)} \right) \]

\[ K_p = \infty \]

\[ K_v = \lim_{{s \to 0}} sG(s) \]

\[ K_v = \lim_{{s \to 0}} \left( \frac{100s(s + 2)(s + 5)}{s^2(s + 8)(s + 12)} \right) \]

\[ K_v = \infty \]

\[ K_a = \lim_{{s \to 0}} s^2G(s) \]

\[ K_a = \lim_{{s \to 0}} \left( \frac{100s^2(s + 2)(s + 5)}{s^2(s + 8)(s + 12)} \right) \]

\[ K_a = \left( \frac{100(0 + 2)(0 + 5)}{(0 + 8)(0 + 12)} \right) = 10.4 \]
Example#8 (Lecture-16-17-18)

Figure (a) shows a mechanical vibratory system. When 2 lb of force (step input) is applied to the system, the mass oscillates, as shown in Figure (b). Determine \( m \), \( b \), and \( k \) of the system from this response curve.
Example#8

Figure (a) shows a mechanical vibratory system. When 2 lb of force (step input) is applied to the system, the mass oscillates, as shown in Figure (b). Determine $m$, $b$, and $k$ of the system from this response curve.

It follows that the steady-state value of $x$ is

$$x(\infty) = \lim_{s \to 0} sX(s) = \frac{2}{k} = 0.1 \text{ ft}$$