Automatic Control Systems (FCS)

Lecture-6
Time Domain Analysis of 2\textsuperscript{nd} Order Systems
Introduction

• We have already discussed the affect of location of pole and zero on the transient response of 1\textsuperscript{st} order systems.

• Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.

• Varying a first-order system's parameters (T, K) simply changes the speed and offset of the response

• Whereas, changes in the parameters of a second-order system can change the \textit{form} of the response.

• A second-order system can display characteristics much like a first-order system or, depending on component values, display damped or \textit{pure oscillations} for its transient response.
Introduction

• A general second-order system (without zeros) is characterized by the following transfer function.

\[ R(s) \rightarrow E(s) \rightarrow \frac{\omega_n^2}{s(s + 2\zeta \omega_n)} \rightarrow C(s) \]

Open-Loop Transfer Function

\[ G(s) = \frac{\omega_n^2}{s(s + 2\zeta \omega_n)} \]

Closed-Loop Transfer Function

\[ \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]
Introduction

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

\(\zeta\) → damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.

\(\omega_n\) → un-damped natural frequency of the second order system, which is the frequency of oscillation of the system without damping.
Example#1

• Determine the un-damped natural frequency and damping ratio of the following second order system.

\[
\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}
\]

• Compare the numerator and denominator of the given transfer function with the general 2\textsuperscript{nd} order transfer function.

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

\[
\omega_n^2 = 4 \quad \Rightarrow \quad \omega_n = 2 \text{ rad/sec}
\]

\[
s^2 + 2\zeta \omega_n s + \omega_n^2 = s^2 + 2s + 4 \quad \Rightarrow \quad 2\zeta \omega_n s = 2s \\
\Rightarrow \quad \zeta \omega_n = 1 \\
\Rightarrow \quad \zeta = 0.5
\]
Introduction

\[ \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_ns + \omega_n^2} \]

- The closed-loop poles of the system are

\[ -\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1} \]
\[ -\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1} \]
Introducing

\[-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}\]
\[-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}\]

- Depending upon the value of \( \zeta \), a second-order system can be set into one of the four categories:

1. **Overdamped** - when the system has two real distinct poles (\( \zeta > 1 \)).

![Diagram showing the complex plane with poles at -c, -b, and -a.]
Introduction

\[-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}\]
\[-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}\]

• According the value of $\zeta$, a second-order system can be set into one of the four categories:

2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)
Introduction

\[-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}\]

\[-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}\]

- According the value of \(\zeta\), a second-order system can be set into one of the four categories:

3. **Undamped** - when the system has two imaginary poles (\(\zeta = 0\)).
Introduction

\[- \omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}\]

\[- \omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}\]

• According the value of $\zeta$, a second-order system can be set into one of the four categories:

4. **Critically damped** - when the system has two real but equal poles ($\zeta = 1$).
Time-Domain Specification

For $0 < \zeta < 1$ and $\omega_n > 0$, the 2nd order system’s response due to a unit step input looks like

![Diagram showing the response of a 2nd order system](image)
Time-Domain Specification

• The delay ($t_d$) time is the time required for the response to reach half the final value the very first time.
Time-Domain Specification

• The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.

• For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.
Time-Domain Specification

• The peak time is the time required for the response to reach the first peak of the overshoot.
Time-Domain Specification

The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

\[
\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\% 
\]

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.
Time-Domain Specification

- The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).
S-Plane

- Natural Undamped Frequency.

- Distance from the origin of s-plane to pole is natural undamped frequency in rad/sec.
S-Plane

• Let us draw a circle of radius 3 in s-plane.

• If a pole is located anywhere on the circumference of the circle the natural undamped frequency would be 3 \text{ rad/sec}.
Therefore the s-plane is divided into Constant Natural Undamped Frequency ($\omega_n$) Circles.
S-Plane

• Damping ratio.

• Cosine of the angle between vector connecting origin and pole and –ve real axis yields damping ratio.

\[ \zeta = \cos \theta \]
S-Plane

- For Underdamped system $0^\circ < \theta < 90^\circ$ therefore, $0 < \zeta < 1$
S-Plane

- For Undamped system $\theta = 90^\circ$ therefore, $\zeta = 0$
S-Plane

- For overdamped and critically damped systems $\theta = 0^\circ$
  therefore, $\zeta \geq 1$
S-Plane

- Draw a vector connecting origin of s-plane and some point $P$. 

\[ \zeta = \cos 45^\circ = 0.707 \]
Therefore, s-plane is divided into sections of constant damping ratio lines.
Example-2

- Determine the natural frequency and damping ratio of the poles from the following pz-map.
Example-3

- Determine the natural frequency and damping ratio of the poles from the given pz-map.

- Also determine the transfer function of the system and state whether the system is underdamped, overdamped, undamped or critically damped.
Example-4

• The natural frequency of closed loop poles of $2^{\text{nd}}$ order system is 2 rad/sec and damping ratio is 0.5.

• Determine the location of closed loop poles so that the damping ratio remains same but the natural undamped frequency is doubled.

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{4}{s^2 + 2s + 4}
\]
Example-4

- Determine the location of closed loop poles so that the damping ratio remains same but the natural undamped frequency is doubled.
S-Plane

\[-\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}\]

\[-\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}\]
Step Response of underdamped System

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

\[
C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}
\]

- The partial fraction expansion of above equation is given as

\[
C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

\[
C(s) = \frac{1}{s + 2\zeta\omega_n} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}
\]
Step Response of underdamped System

\[ C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} \]

• Above equation can be written as

\[ C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \]

• Where \( \omega_d = \omega_n\sqrt{1 - \zeta^2} \), is the frequency of transient oscillations and is called damped natural frequency.

• The inverse Laplace transform of above equation can be obtained easily if \( C(s) \) is written in the following form:

\[ C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \]
Step Response of underdamped System

\[
C(s) = \frac{1}{s} - \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_d^2} - \frac{\zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_d^2}
\]

\[
C(s) = \frac{1}{s} - \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_d^2} - \frac{\zeta}{\sqrt{1 - \zeta^2}} \frac{\omega_n \sqrt{1 - \zeta^2}}{(s + \zeta \omega_n)^2 + \omega_d^2}
\]

\[
C(s) = \frac{1}{s} - \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_d^2} - \frac{\zeta \omega_d}{\sqrt{1 - \zeta^2} (s + \zeta \omega_n)^2 + \omega_d^2}
\]

\[
c(t) = 1 - e^{-\zeta \omega_n t} \cos \omega_d t - \frac{\zeta}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \omega_d t
\]
Step Response of underdamped System

\[ c(t) = 1 - e^{-\zeta \omega_n t} \cos \omega_d t - \frac{\zeta}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \sin \omega_d t \]

\[ c(t) = 1 - e^{-\zeta \omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right] \]

• When \( \zeta = 0 \)

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]

\[ = \omega_n \]

\[ c(t) = 1 - \cos \omega_n t \]
Step Response of underdamped System

\[
c(t) = 1 - e^{-\xi \omega_n t} \left[ \cos \omega_d t + \frac{\xi}{\sqrt{1 - \xi^2}} \sin \omega_d t \right]
\]

if \( \xi = 0.1 \) and \( \omega_n = 3 \) rad/sec

![Graph showing the step response of an underdamped system with \( \xi = 0.1 \) and \( \omega_n = 3 \) rad/sec.](image-url)
Step Response of underdamped System

\[ c(t) = 1 - e^{-\zeta \omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right] \]

if \( \zeta = 0.5 \) and \( \omega_n = 3 \) rad/sec
Step Response of underdamped System

\[ c(t) = 1 - e^{-\zeta \omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right] \]

if \( \zeta = 0.9 \) and \( \omega_n = 3 \text{ rad/sec} \)

![Graph showing the step response of an underdamped system with \( \zeta = 0.9 \) and \( \omega_n = 3 \text{ rad/sec} \).]
Step Response of underdamped System
Step Response of underdamped System

The graph shows the step response of an underdamped system with different values of $\omega_n$:

- $\omega_n = 0.5$ (red line)
- $\omega_n = 1$ (blue line)
- $\omega_n = 1.5$ (black line)
- $\omega_n = 2$ (pink line)
- $\omega_n = 2.5$ (yellow line)

The x-axis represents time, ranging from 0 to 10, and the y-axis represents the response amplitude, ranging from 0 to 1.4.
Time Domain Specifications of Underdamped system

- $M_p$: Maximum overshoot
- $t_d$: Rise time
- $t_r$: Settling time
- $t_p$: Peak time
- $t_s$: Overshoot time

Allowable tolerance: 0.05 or 0.02
Time Domain Specifications (Rise Time)

\[ c(t) = 1 - e^{-\zeta \omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right] \]

Put \( t = t_r \) in above equation

\[ c(t_r) = 1 - e^{-\zeta \omega_n t_r} \left[ \cos \omega_d t_r + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t_r \right] \]

Where \( c(t_r) = 1 \)

\[ 0 = -e^{-\zeta \omega_n t_r} \left[ \cos \omega_d t_r + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t_r \right] \]

\[ -e^{-\zeta \omega_n t_r} \neq 0 \]

\[ 0 = \left[ \cos \omega_d t_r + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t_r \right] \]
Time Domain Specifications (Rise Time)

\[
\left[ \cos \omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_r \right] = 0
\]

The above equation can be rewritten as:

\[
\sin \omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta} \cos \omega_d t_r
\]

\[
\tan \omega_d t_r = -\frac{\sqrt{1-\zeta^2}}{\zeta}
\]

\[
\omega_d t_r = \tan^{-1}\left(-\frac{\sqrt{1-\zeta^2}}{\zeta}\right)
\]
Time Domain Specifications (Rise Time)

\[ \omega_d t_r = \tan^{-1} \left( -\frac{\sqrt{1-\zeta^2}}{\zeta} \right) \]

\[ t_r = \frac{1}{\omega_d} \tan^{-1} \left( -\frac{\omega_n \sqrt{1-\zeta^2}}{\omega_n \zeta} \right) \]

\[ t_r = \frac{\pi - \theta}{\omega_d} \]

\[ \theta = \tan^{-1} \frac{a}{b} \]
Time Domain Specifications (Peak Time)

\[ c(t) = 1 - e^{-\zeta \omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right] \]

- In order to find peak time let us differentiate above equation w.r.t \( t \).

\[ \frac{dc(t)}{dt} = \zeta \omega_n e^{-\zeta \omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right] - e^{-\zeta \omega_n t} \left[ -\omega_d \sin \omega_d t + \frac{\zeta \omega_d}{\sqrt{1 - \zeta^2}} \cos \omega_d t \right] \]

\[ 0 = e^{-\zeta \omega_n t} \left[ \zeta \omega_n \cos \omega_d t + \frac{\zeta^2 \omega_n}{\sqrt{1 - \zeta^2}} \sin \omega_d t + \omega_d \sin \omega_d t - \frac{\zeta \omega_d}{\sqrt{1 - \zeta^2}} \cos \omega_d t \right] \]

\[ 0 = e^{-\zeta \omega_n t} \left[ \zeta \omega_n \cos \omega_d t + \frac{\zeta^2 \omega_n}{\sqrt{1 - \zeta^2}} \sin \omega_d t + \omega_d \sin \omega_d t - \frac{\zeta \omega_n \sqrt{1 - \zeta^2}}{\sqrt{1 - \zeta^2}} \cos \omega_d t \right] \]
Time Domain Specifications (Peak Time)

\[ 0 = e^{-\zeta \omega_n t} \left[ \zeta \omega_n \cos \omega_d t + \frac{\zeta^2 \omega_n}{\sqrt{1 - \zeta^2}} \sin \omega_d t + \omega_d \sin \omega_d t - \frac{\zeta \omega_n \sqrt{1 - \zeta^2}}{\sqrt{1 - \zeta^2}} \cos \omega_d t \right] \]

\[ e^{-\zeta \omega_n t} \left[ \frac{\zeta^2 \omega_n}{\sqrt{1 - \zeta^2}} \sin \omega_d t + \omega_d \sin \omega_d t \right] = 0 \]

\[ e^{-\zeta \omega_n t} \neq 0 \left[ \frac{\zeta^2 \omega_n}{\sqrt{1 - \zeta^2}} \sin \omega_d t + \omega_d \sin \omega_d t \right] = 0 \]

\[ \sin \omega_d t \left[ \frac{\zeta^2 \omega_n}{\sqrt{1 - \zeta^2}} + \omega_d \right] = 0 \]
Time Domain Specifications (Peak Time)

\[ \sin \omega_d t \left[ \frac{\zeta^2 \omega_n}{\sqrt{1 - \zeta^2}} + \omega_d \right] = 0 \]

\[ \left[ \frac{\zeta^2 \omega_n}{\sqrt{1 - \zeta^2}} + \omega_d \right] \neq 0 \quad \text{sin} \omega_d t = 0 \]

\[ \omega_d t = \sin^{-1} 0 \]

\[ t = \frac{0, \pi, 2\pi, \ldots}{\omega_d} \]

- Since for underdamped stable systems first peak is maximum peak therefore,

\[ t_p = \frac{\pi}{\omega_d} \]
Time Domain Specifications (Maximum Overshoot)

Maximum percent overshoot = \( \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\% \)

\[
c(t_p) = 1 - e^{-\zeta \omega_n t_p} \left[ \cos \omega_d t_p + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t_p \right]
\]

\[
c(\infty) = 1
\]

\[
M_p = \left[ \frac{1}{1 - e^{-\zeta \omega_n t_p} \left( \cos \omega_d t_p + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t_p \right)} - 1 \right] \times 100
\]

Put \( t_p = \frac{\pi}{\omega_d} \) in above equation

\[
M_p = -e^{-\zeta \omega_n \frac{\pi}{\omega_d}} \left( \cos \omega_d \frac{\pi}{\omega_d} + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d \frac{\pi}{\omega_d} \right) \times 100
\]
Time Domain Specifications (Maximum Overshoot)

\[ M_p = \left[ -e^{-\zeta \omega_n \frac{\pi}{\omega_d}} \left( \cos \phi_d \frac{\pi}{\omega_d} + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \phi_d \frac{\pi}{\omega_d} \right) \right] \times 100 \]

Put \( \omega_d = \omega_n \sqrt{1-\zeta^2} \) in above equation

\[ M_p = \left[ -e^{-\zeta \omega_n \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}} \left( \cos \pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi \right) \right] \times 100 \]

\[ M_p = \left[ -e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} \left( -1 + 0 \right) \right] \times 100 \]

\[ M_p = e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} \times 100 \]
Time Domain Specifications (Settling Time)

\[ c(t) = 1 - e^{-\zeta \omega_n t} \left[ \cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \right] \]

- **Real Part**: \(-\omega_n \zeta \pm \omega_n \sqrt{\zeta^2 - 1}\)
- **Imaginary Part**: 

Exponential decay generated by real part of complex pole pair

Sinusoidal oscillation generated by imaginary part of complex pole pair
Time Domain Specifications (Settling Time)

- Settling time (2%) criterion
  - Time consumed in exponential decay up to 98% of the input.

\[ t_s = 4T = \frac{4}{\xi \omega_n} \]

- Settling time (5%) criterion
  - Time consumed in exponential decay up to 95% of the input.

\[ t_s = 3T = \frac{3}{\xi \omega_n} \]
## Summary of Time Domain Specifications

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<th>Peak Time</th>
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<td>( t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1 - \zeta^2}} )</td>
<td>( t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} )</td>
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<td>Maximum Overshoot</td>
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<td>( M_p = e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} \times 100 )</td>
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Example#5

- Consider the system shown in following figure, where damping ratio is 0.6 and natural undamped frequency is 5 rad/sec. Obtain the rise time $t_r$, peak time $t_p$, maximum overshoot $M_p$, and settling time 2% and 5% criterion $t_s$ when the system is subjected to a unit-step input.
Example#5

Rise Time

\[ t_r = \frac{\pi - \theta}{\omega_d} \]

Peak Time

\[ t_p = \frac{\pi}{\omega_d} \]

Settling Time (2%)

\[ t_s = 4T = \frac{4}{\zeta \omega_n} \]

Settling Time (4%)

\[ t_s = 3T = \frac{3}{\zeta \omega_n} \]

Maximum Overshoot

\[ M_p = e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} \times 100 \]
Example#5

Rise Time

\[ t_r = \frac{\pi - \theta}{\omega_d} \]

\[ t_r = \frac{3.141 - \theta}{\omega_n \sqrt{1 - \zeta^2}} \]

\[ \theta = \tan^{-1}\left(\frac{\omega_n \sqrt{1 - \zeta^2}}{\zeta \omega_n}\right) = 0.93 \text{ rad} \]

\[ t_r = \frac{3.141 - 0.93}{5\sqrt{1 - 0.6^2}} = 0.55 s \]
Example #5

**Peak Time**

\[ t_p = \frac{\pi}{\omega_d} \]

\[ t_p = \frac{3.141}{4} = 0.785 \text{s} \]

**Settling Time (2%)**

\[ t_s = \frac{4}{\zeta \omega_n} \]

\[ t_s = \frac{4}{0.6 \times 5} = 1.33 \text{s} \]

**Settling Time (4%)**

\[ t_s = \frac{3}{\zeta \omega_n} \]

\[ t_s = \frac{3}{0.6 \times 5} = 1 \text{s} \]
Example#5

Maximum Overshoot

\[ M_p = e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} \times 100 \]

\[ M_p = e^{-\frac{3.141 \times 0.6}{\sqrt{1-0.6^2}}} \times 100 \]
Example#6

• For the system shown in Figure-(a), determine the values of gain $K$ and velocity-feedback constant $K_h$ so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec. With these values of $K$ and $K_h$, obtain the rise time and settling time. Assume that $J=1$ kg-m$^2$ and $B=1$ N-m/rad/sec.
Example #6

\[
\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}
\]
Example #6

\[ \frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K} \]

Since \( J = 1 \text{ kgm}^2 \) and \( B = 1 \text{ Nm/rad/sec} \)

\[ \frac{C(s)}{R(s)} = \frac{K}{s^2 + (1 + KK_h)s + K} \]

- Comparing above T.F with general 2\(^{nd}\) order T.F

\[ \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

\[ \omega_n = \sqrt{K} \quad \zeta = \frac{(1 + KK_h)}{2\sqrt{K}} \]
Example #6

\[ \omega_n = \sqrt{K} \]

- Maximum overshoot is 0.2.

\[ M_p = e^{-\left(\zeta / \sqrt{1 - \zeta^2}\right)\pi} \]
\[ e^{-\left(\zeta / \sqrt{1 - \zeta^2}\right)\pi} = 0.2 \]
\[ -\frac{\pi \zeta}{\sqrt{1 - \zeta^2}} \]
\[ \ln(e^{-\sqrt{1 - \zeta^2}}) = \ln(0.2) \]
\[ \frac{\zeta \pi}{\sqrt{1 - \zeta^2}} = 1.61 \]
\[ \zeta = 0.456 \]

\[ \zeta = \frac{(1 + KK_h)}{2\sqrt{K}} \]

- The peak time is 1 sec

\[ t_p = \frac{\pi}{\omega_d} \]
\[ 1 = \frac{3.141}{\omega_n \sqrt{1 - \zeta^2}} \]
\[ \omega_n = \frac{3.141}{\sqrt{1 - 0.456^2}} \]
\[ \omega_n = 3.53 \]
Example#6

\[ \zeta = 0.456 \quad \omega_n = 3.96 \]

\[ \omega_n = \sqrt{K} \]

\[ 3.53 = \sqrt{K} \]

\[ 3.53^2 = K \]

\[ K = 12.5 \]

\[ \zeta = \frac{1 + K K_h}{2\sqrt{K}} \]

\[ 0.456 \times 2\sqrt{12.5} = (1 + 12.5K_h) \]

\[ K_h = 0.178 \]
Example#6

\[ \zeta = 0.456 \]

\[ \omega_n = 3.96 \]

\[ t_r = \frac{\pi - \theta}{\omega_n \sqrt{1 - \zeta^2}} \]

\[ t_r = 0.65s \]

\[ t_s = \frac{4}{\zeta \omega_n} \]

\[ t_s = 2.48s \]

\[ t_s = \frac{3}{\zeta \omega_n} \]

\[ t_s = 1.86s \]
Example #7

When the system shown in Figure (a) is subjected to a unit-step input, the system output responds as shown in Figure (b). Determine the values of $a$ and $c$ from the response curve.

\[
\frac{1}{s(cs + 1)}
\]
Example#8

Figure (a) shows a mechanical vibratory system. When 2 lb of force (step input) is applied to the system, the mass oscillates, as shown in Figure (b). Determine $m$, $b$, and $k$ of the system from this response curve.
Example #9

Given the system shown in following figure, find \( J \) and \( D \) to yield 20% overshoot and a settling time of 2 seconds for a step input of torque \( T(t) \).

\[
G(s) = \frac{1/J}{s^2 + \frac{D}{J} s + \frac{K}{J}}
\]

\[
\omega_n = \sqrt{\frac{K}{J}}
\]

\[
T_s = 2 = \frac{4}{\zeta \omega_n}
\]

\[
2\zeta \omega_n = 4
\]

\[
\zeta = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}}
\]
Example#9

\[ \omega_n = \sqrt{\frac{K}{J}} \quad \quad \quad \zeta = 2\sqrt{\frac{J}{K}} \]

20\% \text{ overshoot implies } \zeta = 0.456. \text{ Therefore,} \]

\[ \zeta = 2\sqrt{\frac{J}{K}} = 0.456 \]

Hence,

\[ \frac{J}{K} = 0.052 \]

From the problem statement, \( K = 5 \text{ N-m/rad} \).

\[ J = 0.26 \text{ kg-m}^2. \]
Example #9

\[ G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}} \]

\[ 2\zeta \omega_n = \frac{D}{J} \]

\[ D = 1.04 \text{ N-m-s/rad} \]
Step Response of critically damped System ( $\zeta = 1$ )

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \omega_n)^2}$$

The partial fraction expansion of above equation is given as

$$\frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2}$$

$$C(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

$$c(t) = 1 - e^{-\omega_n t} - \omega_n e^{-\omega_n t} t$$

$$c(t) = 1 - e^{-\omega_n t} \left(1 + \omega_n t\right)$$
Step Response of overdamped and undamped Systems

• Home Work
(a) \[ R(s) = \frac{1}{s} \]

\[ G(s) = \frac{b}{s^2 + as + b} \]

General

(b) \[ R(s) = \frac{1}{s} \]

\[ G(s) = \frac{9}{s^2 + 9s + 9} \]

Overdamped

(c) \[ R(s) = \frac{1}{s} \]

\[ G(s) = \frac{9}{s^2 + 2s + 9} \]

Underdamped

(d) \[ R(s) = \frac{1}{s} \]

\[ G(s) = \frac{9}{s^2 + 9} \]

Undamped

(c(t)) \[ c(t) = 1 - e^{-t(\cos \sqrt{8} t + \frac{\sqrt{8}}{8} \sin \sqrt{8} t)} \]

\[ = 1 - 1.06 e^{-t} \cos(\sqrt{8} t - 19.47°) \]

\[ c(t) = 1 - e^{-7.854t} - 1.171e^{-1.146t} \]

\[ c(t) = 1 - \cos 3t \]
Example 10: Describe the nature of the second-order system response via the value of the damping ratio for the systems with transfer function

1. \[ G(s) = \frac{12}{s^2 + 8s + 12} \]

2. \[ G(s) = \frac{16}{s^2 + 8s + 16} \]

3. \[ G(s) = \frac{20}{s^2 + 8s + 20} \]