Control Systems (CS)

Lecture-10-11
Signal Flow Graphs

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Outline

• Introduction to Signal Flow Graphs
  – Definitions
  – Terminologies
  – Examples
• Mason’s Gain Formula
  – Examples
• Signal Flow Graph from Block Diagrams
• Design Examples
Introduction

• Alternative method to block diagram representation, developed by Samuel Jefferson Mason.

• Advantage: the availability of a flow graph gain formula, also called Mason’s gain formula.

• A signal-flow graph consists of a network in which nodes are connected by directed branches.

• It depicts the flow of signals from one point of a system to another and gives the relationships among the signals.
Fundamentals of Signal Flow Graphs

• Consider a simple equation below and draw its signal flow graph:

\[ y = ax \]

• The signal flow graph of the equation is shown below;

\[ x \rightarrow a \rightarrow y \]

• Every variable in a signal flow graph is designed by a **Node**.
• Every transmission function in a signal flow graph is designed by a **Branch**.
• Branches are always **unidirectional**.
• The arrow in the branch denotes the **direction** of the signal flow.
Signal-Flow Graph Models

\[ Y_1(s) = G_{11}(s) \cdot R_1(s) + G_{12}(s) \cdot R_2(s) \]

\[ Y_2(s) = G_{21}(s) \cdot R_1(s) + G_{22}(s) \cdot R_2(s) \]
$r_1 \text{ and } r_2$ are inputs and $x_1$ and $x_2$ are outputs

\[ a_{11} \cdot x_1 + a_{12} \cdot x_2 + r_1 = x_1 \]

\[ a_{21} \cdot x_1 + a_{22} \cdot x_2 + r_2 = x_2 \]
$x_0$ is input and $x_4$ is output

\[
x_1 = ax_0 + bx_1 + cx_2 \\
x_2 = dx_1 + ex_3 \\
x_3 = fx_0 + gx_2 \\
x_4 = hx_3
\]
Construct the signal flow graph for the following set of simultaneous equations.

\[ x_2 = A_{21}x_1 + A_{23}x_3 \quad x_3 = A_{31}x_1 + A_{32}x_2 + A_{33}x_3 \quad x_4 = A_{42}x_2 + A_{43}x_3 \]

- There are four variables in the equations (i.e., \( x_1, x_2, x_3 \), and \( x_4 \)) therefore four nodes are required to construct the signal flow graph.
- Arrange these four nodes from left to right and connect them with the associated branches.

Another way to arrange this graph is shown in the figure.
Terminologies

- An **input node** or source contain only the outgoing branches. i.e., $X_1$
- An **output node** or sink contain only the incoming branches. i.e., $X_4$
- A **path** is a continuous, unidirectional succession of branches along which no node is passed more than ones. i.e.,

  $X_1 \text{ to } X_2 \text{ to } X_3 \text{ to } X_4$

- A **forward path** is a path from the input node to the output node. i.e.,

  $X_1 \text{ to } X_2 \text{ to } X_3 \text{ to } X_4$, and $X_1 \text{ to } X_2 \text{ to } X_4$, are forward paths.

- A **feedback path** or feedback loop is a path which originates and terminates on the same node. i.e.; **$X_2 \text{ to } X_3 \text{ and back to } X_2$** is a feedback path.
Terminologies

- A **self-loop** is a feedback loop consisting of a single branch. i.e.; $A_{33}$ is a self loop.

- The **gain** of a branch is the transmission function of that branch.

- The **path gain** is the product of branch gains encountered in traversing a path. i.e. the gain of forwards path $X_1 \text{ to } X_2 \text{ to } X_3 \text{ to } X_4$ is $A_{21}A_{32}A_{43}$.

- The **loop gain** is the product of the branch gains of the loop. i.e., the loop gain of the feedback loop from $X_2$ to $X_3$ and back to $X_2$ is $A_{32}A_{23}$.

- Two loops, paths, or loop and a path are said to be **non-touching** if they have no nodes in common.
Consider the signal flow graph below and identify the following:

a) Input node.
b) Output node.
c) Forward paths.
d) Feedback paths (loops).
e) Determine the loop gains of the feedback loops.
f) Determine the path gains of the forward paths.
g) Non-touching loops
Consider the signal flow graph below and identify the following:

- There are two forward path gains;

1. \( G_1(s)G_2(s)G_3(s)G_4(s)G_5(s)G_7(s) \)
2. \( G_1(s)G_2(s)G_3(s)G_4(s)G_6(s)G_7(s) \)
Consider the signal flow graph below and identify the following:

- There are four loops:
  1. $G_2(s)H_1(s)$
  2. $G_4(s)H_2(s)$
  3. $G_4(s)G_5(s)H_3(s)$
  4. $G_4(s)G_6(s)H_3(s)$
Consider the signal flow graph below and identify the following:

- Nontouching loop gains:

1. \([G_2(s)H_1(s)][G_4(s)H_2(s)]\)
2. \([G_2(s)H_1(s)][G_4(s)G_5(s)H_3(s)]\)
3. \([G_2(s)H_1(s)][G_4(s)G_6(s)H_3(s)]\)
Consider the signal flow graph below and identify the following

a) Input node.
b) Output node.
c) Forward paths.
d) Feedback paths.
e) Self loop.
f) Determine the loop gains of the feedback loops.
g) Determine the path gains of the forward paths.
Input and output Nodes

a) Input node \( X_1 \)

b) Output node \( X_8 \)
(c) Forward Paths

$X_1$ to $X_2$ to $X_3$ to $X_4$ to $X_5$ to $X_6$ to $X_7$ to $X_8$

$X_1$ to $X_2$ to $X_7$ to $X_8$

$X_1$ to $X_2$ to $X_4$ to $X_5$ to $X_6$ to $X_7$ to $X_8$
(d) Feedback Paths or Loops

- $X_2$ to $X_3$ to $X_2$
- $X_3$ to $X_4$ to $X_3$
- $X_4$ to $X_5$ to $X_4$
- $X_5$ to $X_6$ to $X_5$
- $X_6$ to $X_7$ to $X_6$
(d) Feedback Paths or Loops

X_2 to X_4 to X_3 to X_2

X_5 to X_6 to X_7 to X_5
(d) Feedback Paths or Loops

$X_2$ to $X_7$ to $X_5$ to $X_4$ to $X_3$ to $X_2$
(d) Feedback Paths or Loops

$X_2$ to $X_7$ to $X_6$ to $X_5$ to $X_4$ to $X_3$ to $X_2$
(e) Self Loop(s)
(f) Loop Gains of the Feedback Loops

\[
\begin{align*}
A_{32} A_{23} & \quad A_{76} A_{67}; \\
A_{43} A_{34} & \quad A_{65} A_{76} A_{57}; \\
A_{54} A_{45} & \quad A_{77}; \\
A_{65} A_{56} & \quad A_{42} A_{34} A_{23}; \\
A_{72} A_{57} A_{45} A_{34} A_{23}; \\
A_{72} A_{67} A_{56} A_{45} A_{34} A_{23}
\end{align*}
\]
(g) Path Gains of the Forward Paths

\[ A_{32} A_{43} A_{54} A_{65} A_{76} \]

\[ A_{72} \]

\[ A_{42} A_{54} A_{65} A_{76} \]
Mason’s Rule (Mason, 1953)

• The block diagram reduction technique requires successive application of fundamental relationships in order to arrive at the system transfer function.

• On the other hand, Mason’s rule for reducing a signal-flow graph to a single transfer function requires the application of one formula.

• The formula was derived by S. J. Mason when he related the signal-flow graph to the simultaneous equations that can be written from the graph.
Mason’s Rule:

• The transfer function, \( \frac{C(s)}{R(s)} \), of a system represented by a signal-flow graph is:

\[
\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta}
\]

Where

\( n \) = number of forward paths.
\( P_i \) = the \( i^{th} \) forward-path gain.
\( \Delta \) = Determinant of the system
\( \Delta_i \) = Determinant of the \( i^{th} \) forward path

• \( \Delta \) is called the signal flow graph determinant or characteristic function. Since \( \Delta=0 \) is the system characteristic equation.
Mason’s Rule:

\[
\frac{C(s)}{R(s)} = \sum_{i=1}^{n} \frac{P_i \Delta_i}{\Delta}
\]

\[\Delta = 1 - \text{(sum of all individual loop gains)} + \text{(sum of the products of the gains of all possible two loops that do not touch each other)} - \text{(sum of the products of the gains of all possible three loops that do not touch each other)} + \ldots \text{ and so forth with sums of higher number of non-touching loop gains}\]

\[\Delta_i = \text{value of } \Delta \text{ for the part of the block diagram that does not touch the } i\text{-th forward path } (\Delta_i = 1 \text{ if there are no non-touching loops to the } i\text{-th path})\]
Systematic approach

1. Calculate forward path gain $P_i$ for each forward path $i$.
2. Calculate all loop transfer functions
3. Consider non-touching loops 2 at a time
4. Consider non-touching loops 3 at a time
5. etc
6. Calculate $\Delta$ from steps 2, 3, 4 and 5
7. Calculate $\Delta_i$ as portion of $\Delta$ not touching forward path $i$
Example #1: Apply Mason’s Rule to calculate the transfer function of the system represented by following Signal Flow Graph

There are two forward paths:

\[ P_1 = G_1G_2G_4 \quad P_2 = G_1G_3G_4 \]

Therefore,

\[
\frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}
\]

There are three feedback loops

\[ L_1 = G_1G_4H_1, \quad L_2 = -G_1G_2G_4H_2, \quad L_3 = -G_1G_3G_4H_2 \]
Example#1: Apply Mason’s Rule to calculate the transfer function of the system represented by following Signal Flow Graph

There are no non-touching loops, therefore

\[ \Delta = 1 - \text{ (sum of all individual loop gains) } \]

\[ \Delta = 1 - \left( L_1 + L_2 + L_3 \right) \]

\[ \Delta = 1 - \left( G_1 G_4 H_1 - G_1 G_2 G_4 H_2 - G_1 G_3 G_4 H_2 \right) \]
Example#1: Apply Mason’s Rule to calculate the transfer function of
the system represented by following Signal Flow Graph

\[ \Delta_1 = 1 - (\text{sum of all individual loop gains}) + \ldots \]
\[ \Delta_1 = 1 \]

Eliminate forward path-1

Eliminate forward path-2

\[ \Delta_2 = 1 - (\text{sum of all individual loop gains}) + \ldots \]
\[ \Delta_2 = 1 \]
Example#1: Continue

\[
\frac{C}{R} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta} = \frac{G_1G_2G_4 + G_1G_3G_4}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}
\]

\[
= \frac{G_1G_4(G_2 + G_3)}{1 - G_1G_4H_1 + G_1G_2G_4H_2 + G_1G_3G_4H_2}
\]
Example#2: Apply Mason’s Rule to calculate the transfer function of the system represented by following Signal Flow Graph

1. Calculate forward path gains for each forward path.

   \[ P_1 = G_1G_2G_3G_4 \text{ (path 1)} \quad \text{and} \quad P_2 = G_5G_6G_7G_8 \text{ (path 2)} \]

2. Calculate all loop gains.

   \[ L_1 = G_2H_2, \quad L_2 = H_3G_3, \quad L_3 = G_6H_6, \quad L_4 = G_7H_7 \]

3. Consider two non-touching loops.

   \[ L_1L_3, \quad L_1L_4, \quad L_2L_4, \quad L_2L_3 \]
4. Consider three non-touching loops.
   None.

5. Calculate $\Delta$ from steps 2,3,4.

\[
\Delta = 1 - \left( L_1 + L_2 + L_3 + L_4 \right) + \left( L_1 L_3 + L_1 L_4 + L_2 L_3 + L_2 L_4 \right)
\]

\[
\Delta = 1 - \left( G_2 H_2 + H_3 G_3 + G_6 H_6 + G_7 H_7 \right) + \\
\left( G_2 H_2 G_6 H_6 + G_2 H_2 G_7 H_7 + H_3 G_3 G_6 H_6 + H_3 G_3 G_7 H_7 \right)
\]
Example#2: continue

Eliminate forward path-1

\[ \Delta_1 = 1 - \left( L_3 + L_4 \right) \]
\[ \Delta_1 = 1 - \left( G_6 H_6 + G_7 H_7 \right) \]

Eliminate forward path-2

\[ \Delta_2 = 1 - \left( L_1 + L_2 \right) \]
\[ \Delta_2 = 1 - \left( G_2 H_2 + G_3 H_3 \right) \]
Example#2: continue

\[ \frac{Y(s)}{R(s)} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta} \]

\[ \frac{Y(s)}{R(s)} = \frac{G_1 G_2 G_3 G_4 [1 - (G_6 H_6 + G_7 H_7)] + G_3 G_6 G_7 G_8 [1 - (G_2 H_2 + G_3 H_3)]}{1 - (G_2 H_2 + G_3 G_3 + G_6 H_6 + G_7 H_7)} + (G_2 H_2 G_6 H_6 + G_2 H_2 G_7 H_7 + H_3 G_3 G_6 H_6 + H_3 G_3 G_7 H_7) } \]
Example#3

• Find the transfer function, \( C(s)/R(s) \), for the signal-flow graph in figure below.

\[ R(s) \xrightarrow{G_1(s)} V_4(s) \xrightarrow{H_1(s)} V_6(s) \xrightarrow{G_8(s)} V_5(s) \xrightarrow{H_4(s)} C(s) \]

\[ G_1(s) \quad G_2(s) \quad G_3(s) \quad G_4(s) \quad G_5(s) \]

\[ V_4(s) \quad V_3(s) \quad V_2(s) \quad V_1(s) \]

\[ H_1(s) \quad H_2(s) \quad G_7(s) \quad G_6(s) \]
Example#3

- There is only one forward Path.

\[ P_1 = G_1(s)G_2(s)G_3(s)G_4(s)G_5(s) \]
Example #3

- There are four feedback loops.

\[ L_1: G_2(s)H_1(s) \]
\[ L_2: G_4(s)H_2(s) \]
\[ L_3: G_7(s)H_4(s) \]
\[ L_4: G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s) \]
Example#3

- Non-touching loops taken two at a time.
Example#3

• Non-touching loops taken three at a time.

\[ L_1, L_2, L_3: G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s) \]
Example #3

\[ \Delta_1 = 1 - G_7(s)H_4(s) \]

\[ \Delta = 1 - [G_2(s)H_1(s) + G_4(s)H_2(s) + G_7(s)H_4(s) + G_2(s)G_3(s)G_4(s)G_5(s)G_6(s)G_7(s)G_8(s)] + [G_2(s)H_1(s)G_4(s)H_2(s) + G_2(s)H_1(s)G_7(s)H_4(s) + G_4(s)H_2(s)G_7(s)H_4(s)] - [G_2(s)H_1(s)G_4(s)H_2(s)G_7(s)H_4(s)] \]
Example #4: Apply Mason’s Rule to calculate the transfer function of the system represented by following Signal Flow Graph

There are three forward paths, therefore $n=3$.

\[
\frac{C(s)}{R(s)} = \frac{\sum_{i=1}^{3} P_i \Delta_i}{\Delta} = \frac{P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3}{\Delta}
\]
Example#4: Forward Paths

\[ P_3 = A_{42}A_{54}A_{65}A_{76} \]

\[ P_1 = A_{32}A_{43}A_{54}A_{65}A_{76} \]

\[ P_2 = A_{72} \]
Example#4: Loop Gains of the Feedback Loops

$L_1 = A_{32}A_{23}$
$L_2 = A_{43}A_{34}$
$L_3 = A_{54}A_{45}$
$L_4 = A_{65}A_{56}$

$L_5 = A_{76}A_{67}$
$L_6 = A_{77}$
$L_7 = A_{42}A_{34}A_{23}$
$L_8 = A_{65}A_{76}A_{67}$

$L_9 = A_{72}A_{57}A_{45}A_{34}A_{23}$
$L_{10} = A_{72}A_{67}A_{56}A_{45}A_{34}A_{23}$
Example#4: two non-touching loops

\[ L_1L_3 \quad L_2L_4 \quad L_3L_5 \quad L_4L_6 \quad L_5L_7 \quad L_7L_8 \]

\[ L_1L_4 \quad L_2L_5 \quad L_3L_6 \quad L_4L_7 \]

\[ L_1L_5 \quad L_2L_6 \]

\[ L_1L_6 \quad L_2L_8 \]

\[ L_1L_8 \]
Example#4: Three non-touching loops

\[ L_1L_3 \quad L_2L_4 \quad L_3L_5 \quad L_4L_6 \quad L_5L_7 \quad L_7L_8 \]

\[ L_1L_4 \quad L_2L_5 \quad L_3L_6 \quad L_4L_7 \]

\[ L_1L_5 \quad L_2L_6 \]

\[ L_1L_6 \quad L_2L_8 \]

\[ L_1L_8 \]
From Block Diagram to Signal-Flow Graph Models

Example#5

\[ R(s) \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow G_4 \rightarrow C(s) \]

\[ E(s) \rightarrow X_1 \rightarrow G_2 \rightarrow X_2 \rightarrow G_3 \rightarrow X_3 \rightarrow G_4 \rightarrow C(s) \]

\[ H_1 \rightarrow H_2 \rightarrow H_3 \]

\[ R(s), E(s), X_1, X_2, X_3, C(s) \]
From Block Diagram to Signal-Flow Graph Models
Example#5

\[ \Delta = 1 + (G_1G_2G_3G_4H_3 + G_2G_3H_2 + G_3G_4H_1) \]

\[ P_1 = G_1G_2G_3G_4 ; \quad \Delta_1 = 1 \]

\[ G = \frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1+G_1G_2G_3G_4H_3 + G_2G_3H_2 + G_3G_4H_1} \]
Example#6
Example#6

7 loops:

\[
\begin{align*}
& [G_1 \cdot (-1)]; \quad [G_2 \cdot (-1)]; \quad [G_1 \cdot (-1) \cdot G_2 \cdot 1]; \quad [(-1) \cdot G_1 \cdot 1 \cdot (-1)]; \\
& [(-1) \cdot G_1 \cdot (-1) \cdot G_2 \cdot 1 \cdot (-1)]; \quad [1 \cdot G_2 \cdot 1 \cdot (-1)]; \quad [1 \cdot G_2 \cdot 1 \cdot G_1 \cdot 1 \cdot (-1)].
\end{align*}
\]

3 ‘2 non-touching loops’ :

\[
\begin{align*}
& [G_1 \cdot (-1)] \cdot [G_2 \cdot (-1)]; \quad [(-1) \cdot G_1 \cdot 1 \cdot (-1)] \cdot [G_2 \cdot (-1)]; \\
& [1 \cdot G_2 \cdot 1 \cdot (-1)] \cdot [G_1 \cdot (-1)].
\end{align*}
\]
Then:

\[ \Delta = 1 + 2G_2 + 4G_1G_2 \]

4 forward paths:

\[ p_1 = (-1) \cdot G_1 \cdot 1 \quad \Delta_1 = 1 + G_2 \]
\[ p_2 = (-1) \cdot G_1 \cdot (-1) \cdot G_2 \cdot 1 \quad \Delta_2 = 1 \]
\[ p_3 = 1 \cdot G_2 \cdot 1 \quad \Delta_3 = 1 + G_1 \]
\[ p_4 = 1 \cdot G_2 \cdot 1 \cdot G_1 \cdot 1 \quad \Delta_4 = 1 \]
We have

\[
\frac{C(s)}{R(s)} = \sum \frac{p_k \Delta_k}{\Delta} = \frac{G_2 - G_1 + 2G_1 G_2}{1 + 2G_2 + 4G_1 G_2}
\]
**Example-7:** Determine the transfer function C/R for the block diagram below by signal flow graph techniques.

- The signal flow graph of the above block diagram is shown below.

- There are two forward paths. The path gains are
  \[ P_1 = G_1 G_2 G_3 \text{ and } P_2 = G_4 \]

- The three feedback loop gains are
  \[ P_{11} = -G_2 H_1, \ P_{21} = G_1 G_2 H_1, \ P_{31} = -G_2 G_3 H_2. \]

- No loops are non-touching, hence
  \[ \Delta = 1 - (P_{11} + P_{21} + P_{31}) \]

- Because the loops touch the nodes of P1, hence \( \Delta_1 = 1 \)

- Since no loops touch the nodes of P2, therefore \( \Delta_2 = \Delta \)

- Hence the control ratio \( T = C/R \) is

\[
T = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta} = \frac{G_1 G_2 G_3 + G_4 + G_2 G_4 H_1 - G_1 G_2 G_4 H_1 + G_2 G_3 G_4 H_2}{1 + G_2 H_1 - G_1 G_2 H_1 + G_2 G_3 H_2}
\]
Example-6: Find the control ratio C/R for the system given below.

- The signal flow graph is shown in the figure.

- The two forward path gains are \( P_1 = G_1G_2G_3 \) and \( P_2 = G_1G_4 \).

- The five feedback loop gains are
  \[
  P_{11} = G_1G_2H_1, \quad P_{21} = G_2G_3H_2, \quad P_{31} = -G_1G_2G_3, \\
  P_{41} = G_4H_2, \quad \text{and} \quad P_{51} = -G_1G_4.
  \]

- All feedback loops touches the two forward paths, hence \( \Delta_1 = \Delta_2 = 1 \).

- Hence the control ratio \( \frac{C}{R} = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2G_3 - G_1G_2H_1 - G_2G_3H_2 - G_4H_2 + G_1G_4} \).
Design Example#1

\[ V_1(s) = \frac{1}{Cs} I_1(s) + I_1(s)R \]
\[ V_2(s) = I_1(s)R \]

\[ CsV_1(s) - CsV_2(s) = I_1(s) \]
Design Example #2

\[ F = M_1 s^2 X_1 + k_1 (X_1 - X_2) \quad 0 = M_2 s^2 X_2 + k_1 (X_2 - X_1) + k_2 X_2 \]

(i) \[ F + k_1 X_2 = (M_1 s^2 + f_1 s + k_1) X_1 \]

(ii) \[ k_1 X_1 = (M_2 s^2 + f_2 s + k_1 + k_2) X_2 \]

There are three variables: \( X_1, \ X_2, \) and \( F. \) Therefore we need three nodes.

(iii) \[ \left( \frac{1}{A} \right) F + \left( \frac{k_1}{A} \right) X_2 = X_1 \]

(iv) \[ \left( \frac{k_1}{B} \right) X_1 = X_2 \]
Design Example#2

(iii) \( \left( \frac{1}{A} \right) F + \left( \frac{k_1}{A} \right) X_2 = X_1 \)

(iv) \( \left( \frac{k_1}{B} \right) X_1 = X_2 \)
The forward path gain is \( P_1 = k_1/AB \). The feedback loop gain is \( P_{11} = k_1^2/AB \). Then \( \Delta = 1 - P_{11} = (AB - k_1^2)/AB \) and \( \Delta_1 = 1 \). Finally,

\[
\frac{X_2}{F} = \frac{P_1 \Delta_1}{\Delta} = \frac{k_1}{AB - k_1^2} = \frac{k_1}{(M_1 s^2 + f_1 s + k_1)(M_2 s^2 + f_2 s + k_1 + k_2) - k_1^2}
\]
To download this lecture visit
http://imtiazhussainkalwar.weebly.com/

END OF LECTURES-10-11