From our previous studies, it is clear that:

\[ \nabla \times E = 0 \]
\[ \nabla \times H = J \]
\[ \nabla \cdot D = \rho \]
\[ \nabla \cdot B = 0 \]

**Maxwell’s equations in static case**

A new concept will be introduced:

- The electric field strength, \( E \) produced by changing magnetic field strength, \( H \). **Faraday’s law**
- The magnetic field strength, \( H \) produced by changing electric field strength, \( E \). **Amper’s law**

This is done experimentally by **Faraday** and theoretical efforts of **Maxwell**.

**Faraday’s Law**: (by experimental work)

If a conductor moves in a magnetic field or the magnetic field changes, there will be electromotive force, e.m.f. or voltage arises in the conductor.

Faraday’s law stated as:

\[ e.m.f = -\frac{d}{dt} \phi \]  \[ \text{[V]} \]  \( (1) \)

This equation implies a closed path.

**e.m.f.** in such a direction produces current has flux if added to the original flux, this will reduce the magnitude of e.m.f. (this statement, induced voltage produces opposing flux known as **Lenz’s law**).
If the closed path taken by N turns conductor, then,

\[ e.m.f = -N \frac{d\phi}{dt} \]  \hspace{1cm} \text{[V]}  \hspace{1cm} \text{(2)}

We define:

\[ e.m.f = \int C E.dl \]  \hspace{1cm} \text{[V]}  \hspace{1cm} \text{(3)}

Also, magnetic flux

\[ \phi = \int B.dS \]

Then,

\[ \int C E.dl = -\frac{\partial}{\partial t} \int S B.dS \]  \hspace{1cm} \text{(4)}

This equation is the integral form of 1\textsuperscript{st} Maxwell’s equation.

**Apply Stokes theorem:**

\[ \int C E.dl = \int S (\nabla \times E).dS \]

Eq\textsuperscript{n} (4) becomes:

\[ \int S (\nabla \times E).dS = -\frac{\partial}{\partial t} \int S B.dS \]

i.e.

\[ \nabla \times E = -\frac{\partial}{\partial t} B \]  \hspace{1cm} \text{(5)}

This is 1\textsuperscript{st} Maxwell’s equation in differential or point form.

Which means that a time-changing magnetic field \( B(t) \) produces an electric field, \( E \), it has a property of circulation, its line integral about a general closed path isn’t zero.
If B is not function of time (i.e. static form), then:

\[ \int_E E \cdot dl = 0 \quad \text{and} \quad \nabla \times E = 0 \]

As denoted above.

- **Continuity Equation:**

  If we consider a region bounded by a closed surface s, the current through the closed surface is:
  \[ I = \int_S J \cdot dS \]

  If the charge inside a closed surface is denoted by \( Q_i \), then the rate of decreasing is
  \[ -\frac{d}{dt} Q_i \]

  and the principle of conservation of charge requires:
  \[
  I = \int_S J \cdot dS = -\frac{d}{dt} Q_i = -\frac{d}{dt} \int_V \rho \cdot dV
  \]

  (6)

  Equation (6) is the continuity equation in integral form.

  **By using divergence theorem:**

  \[
  \int_S J \cdot dS = \int_V (\nabla \cdot J) \cdot dV
  \]

  (7)
\[ \therefore (\nabla \cdot J) = -\frac{\partial}{\partial t} \rho_v \]  

Equation (8) is the continuity equation of current in point form.

- **Displacement Current:**
  - **conduction current** occurs in the presence of electric field \( E \) within a conductor of fixed cross section with conductivity \( \sigma \) where:
    \[ J_C = \sigma E \]  
    (9)
  - also **displacement current** occurs within a dielectric material,
    \[ J_d = \frac{\partial}{\partial t} D \]  
    (10)
  - some materials have both currents, \( J_C \) and \( J_d \)

- You know at steady state, for magnetic field \( H \), Ampere’s circuital law is:
  \[ \nabla \times H = J_C \]  
  (11)

Taking divergence, then:

\[ \nabla \cdot (\nabla \times H) = \nabla \cdot J_C \]  
(12)

\[ 0 = \nabla \cdot J_C \]

But we have eqn (8):

\[ \nabla \cdot J = -\frac{\partial}{\partial t} \rho_v \]

so we must add term \( G \) to eqn. 11, then

\[ \nabla \times H = J_C + G \]  
(13)
By taking \( \nabla \).

So,

\[
\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J}_C + \nabla \cdot \mathbf{G}
\]

Then,

\[
0 = \nabla \cdot \mathbf{J}_C + \nabla \cdot \mathbf{G}
\]

\[
\nabla \cdot \mathbf{G} = -\nabla \cdot \mathbf{J} = \frac{\partial}{\partial t} \rho
\]

(14)

but we have Gauss law:

\[
\nabla \cdot \mathbf{D} = \rho
\]

\[
\therefore \nabla \cdot \mathbf{G} = \frac{\partial}{\partial t} \nabla \cdot \mathbf{D}
\]

i.e

\[
\nabla \cdot \mathbf{G} = \nabla \cdot \frac{\partial}{\partial t} \mathbf{D}
\]

then

\[
\mathbf{G} = \frac{\partial}{\partial t} \mathbf{D}
\]

(15)

Eq \( n \) (13) becomes:

\[
\nabla \times \mathbf{H} = \mathbf{J}_C + \frac{\partial}{\partial t} \mathbf{D}
\]

(16)

That’s the second Maxwell’s equation (Amper’s circuital law in point form).

**Apply Stokes theorem:**

\[
\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}
\]

Eq \( n \) (16) becomes:
That’s the second Maxwell’s equation (Amper's circuital law in integral form).

- **Maxwell’s Equation in point form:**
  
  \[
  \nabla X E = -\frac{\partial}{\partial t} B \quad \text{Faraday’s Law}
  \]
  
  \[
  \nabla X H = J_C + \frac{\partial}{\partial t} D \quad \text{Amper’s Law}
  \]
  
  \[
  \nabla . D = \rho_V \quad \text{Gaussian Law, for electric}
  \]
  
  \[
  \nabla . B = 0 \quad \text{Gaussian Law, for electric}
  \]

These four equations are the basic of electromagnetic theory.

They are partial differential equations related E & H to each other, to their sources (charges and current density).

There are some auxiliary equations:

\[
J_C = \sigma E, \quad D = \varepsilon E, \quad B = \mu H
\]

- **Maxwell’s Equations in integral form:**

  we have:

  1. \[
  \nabla X E = -\frac{\partial}{\partial t} B \quad \text{Faraday’s Law in pt. form}
  \]
By using Stokes theorem:
\[ \oint_C \mathbf{E} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} \]

2. \[ \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \]

Faraday’s Law in intg. form

\[ \nabla \times \mathbf{E} = \mathbf{J}_C + \frac{\partial}{\partial t} \mathbf{D} \]

Amper’s Law in pt. form

By using Stokes theorem:
\[ \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} \]

\[ \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S (\mathbf{J}_C + \frac{\partial}{\partial t} \mathbf{D}) \cdot d\mathbf{S} \]

Amper’s Law in intg. Form

3,4- By using Divergence theorem,
- Gauss law for electricity is:
\[ \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_v dV = Q_{en} \]

Gauss law in integral form

- Gauss law for magnetic is:
\[ \oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \]

Gauss law in integral form

These equations are used to determine B.C. (tgt & normal components of fields, E,D,H, and B) between two media.
Meaning of Maxwell’s equations:

1- The first law states that e.m.f around a closed path is equal to the inflow of magnetic current through any surface bounded by the path.

2- The second law states that magnetomotive force m.m.f. around a closed path is equal to the sum of electric displacement and, conduction currents through any surface bounded by the path.

3- The third law states that the total electric displacement flux passing through a closed surface (Gaussian surface) is equal to the total charge inside the surface.

4- The fourth law states that the total magnetic flux passing through any closed surface is zero.

\[
H = \text{magnetic field strength, [A/m]}
\]
\[
D = \text{electric flux density, [C/m]}
\]
\[
\frac{\partial}{\partial t} D = \text{displacement current density, [A/m}^2]\]
\[
J = \text{conduction current density, [A/m}^2]\]
\[
E = \text{electric field [V/m]}
\]
\[
B = \text{magnetic flux density, [wb/m}^2]\text{ or Tesla}
\]
\[
\frac{\partial}{\partial t} B = \text{time derivative of magnetic flux density, [wb/(m}^2.\text{sec})], or}
\]
\[
\text{Tesla/sec}
\]

**Boundary conditions are:**

\[E_{n1} = E_{n2}: \text{tgt component of electric field strength is continuous on the interface between 2 media.}\]

\[D_{n1} - D_{n2} = \rho: \text{normal component of electric flux density equal the charge on the surface between 2 media.}\]
\( \mathbf{H}_{i1} = \mathbf{H}_{i2} \): tgt component of magnetic field strength is continuous on the interface between 2 media.

\( \mathbf{B}_{n1} = \mathbf{B}_{n2} \) normal component of magnetic flux density is continuous on the interface between 2 media.

**properties of the medium:**
- Properties of the medium is characterized by parameters such as \( \varepsilon, \mu, \sigma \)
- Medium is classified into: linear, isotropic, and homogeneous medium where:
  1. **Linear** medium has \( \varepsilon, \mu, \sigma \) are not \( f^n \) of \( \mathbf{E} & \mathbf{H} \)
  2. **Isotropic** medium has \( \mathbf{J} \parallel \mathbf{E} & \mathbf{B} \parallel \mathbf{H} \) and \( \mathbf{D} \parallel \mathbf{B} \)
  3. **Homogeneous** medium has \( \varepsilon, \mu, \sigma \) are constant and not \( f^n \) of \( \text{coodt.} \)
  4. **Free space** has neither charges nor current, it has \( \varepsilon_0, \mu_0 \)

**Hint:**
- From gauss’s law in electric field, we have:
  \[
  \int_S \mathbf{D}.dS = Q = \int_V \rho_V dV
  \]
  apply divergence theorem, \[
  \int_V (\nabla \mathbf{D}).dV = \int_V \rho_V dV
  \]
  we get: ++++++++++++++++++++++
  \[
  \therefore \ \nabla \mathbf{D} = \rho_V
  \]
- From gauss’s law in magnetic field, we have:
\[ \int_S B \cdot dS = 0 \]

apply divergence theorem, \[ \int_V (\nabla \cdot B) \, dV = 0 \]

\[ \therefore \nabla \cdot B = 0 \]
Maxwell’s equations

- **Amper’s Law**:

  \[ \int_{C} H \cdot dl = \int_{S} (J_{C} + \frac{\partial}{\partial t} D) \cdot dS \]  
  integral form

  **By using Stokes theorem**:

  \[ \int_{C} H \cdot dl = \int_{S} (\nabla \times H) \cdot dS \]

  then, \( \nabla \times H = J_{C} + \frac{\partial}{\partial t} D \)  
  point form

- **Faraday’s Law**:

  \[ \int_{C} E \cdot dl = -\frac{\partial}{\partial t} \int_{S} B \cdot dS \]  
  integral form

  **apply Stokes theorem**:

  \[ \int_{C} E \cdot dl = \int_{S} (\nabla \times E) \cdot dS \]

  Then, \( \nabla \times E = -\frac{\partial}{\partial t} B \)  
  point form

- **Gauss Law**:

  a- For electric field

  \[ \int_{S} D \cdot dS = Q = \int_{V} \rho_{V} dV \]  
  integral form

  **apply divergence theorem**:

  \[ \int_{V} (\nabla \cdot D) dV = \int_{V} \rho_{V} dV \]

  then, \( \nabla \cdot D = \rho_{V} \)  
  point form

  b- For magnetic field

  \[ \int_{S} B \cdot dS = 0 \]  
  integral form

  **apply divergence theorem**:

  \[ \int_{V} (\nabla \cdot B) dV = 0 \]

  then, \( \nabla \cdot B = 0 \)  
  point form
Microwave Engineering
Sheet #1
( 4/10/2015)

Time Varying Fields

Q1: A circular loop of 10 cm radius is located in the xy plane in B field given by:

\[ B = (0.5 \cos 377 t)(3 \mathbf{a}_y + 4 \mathbf{a}_z) T. \]

**Determine:** the voltage induced in the loop?

Q2: Find the displacement current density for:

- Next to your radio where the local AM station provides a field strength of

\[ E = 0.02 \sin [0.1927 (3 \times 10^8 t - z)] \mathbf{a}_x \]

- In a good conductor where \( \sigma = 10^7 \text{S/m} \) and the conduction current density is high, like

\[ 10^7 \sin (120 \pi t) \mathbf{a}_x \text{A/m}^2. \]

Q3: An inductor is formed by winding 10 N turns of a thin wire around a wooden rod which has a radius of 2 cm. If a uniform, sinusoidal magnetic field with magnitude 0.01 wb/m\(^2\) and frequency of 10 KHz is directed along the axis of the rod. **Determine:** the voltage induced between the two ends of the wire assuming the two ends are closed together?

Q4: A material having a conductivity \( \sigma \) and permittivity \( \varepsilon \) is placed in a sinusoidal, time-varying electric field having a frequency \( \omega \). **At what** frequency will the conduction current equal to the displacement current? If, \( \sigma = 10^{-12} \text{S/m} \) , and \( \varepsilon = 3\varepsilon_0 \).

Q5: Show that the fields:

\[ E = E_m \sin X \sin t \mathbf{a}_y, \quad H = (E_m / \mu) \cos X \cos t \mathbf{a}_z \]

in free space satisfy Faraday’s law and the two laws of Gauss but don’t satisfy Ampere’s law.

Q6: If \( E \) of radio broadcast signal at T.V Rx is given by:

\[ E = 5 \cos(\omega t - \beta y) \mathbf{a}_z \]

**Determine:** the displacement current density.

If the same field exists in a medium whose conductivity \( \sigma \) is given by:

\[ \sigma = \]
2*10^3 [\Omega^{-1}/cm], **Find:** the conduction current density?

**Q8:** Given \( E = 10 \sin (\omega t - \beta z) a_y \) [v/m] in free space, **Find:** \( D, B \) and \( H \)

**Q9:** a parallel plate capacitor with plate area of 5 cm\(^2\) and plate separation of 3 mm has a voltage 50 sin 10\(^3\)t [v] applied to its plates. **Calculate:** the displacement current assuming \( \varepsilon = 2\varepsilon_o \)

**Q10:** **Show that** the following fields vector in free space satisfy all Maxwell’s equations, \( E = E_0 \cos(\omega t - \beta z) a_x \), \( H = \frac{E_0}{\eta} \cos(\omega t - \beta z) a_y \)

**Q11:** A perfectly conducting sphere of radius \( R \) in free space has a charge \( Q \) uniformly distributed over its surface, utilizing B.C. **Determine** the electric field \( E \) at the surface of the sphere, **show that** by using Gauss law, the result is correct.

---

Good luck                Dr. M.A. Motawea
Solution

Q1: \( \mathbf{B} = (0.5 \cos 377 \, t)(3 \mathbf{a}_y + 4 \mathbf{a}_z) \, T \)

\[
\text{emf} = -N \frac{d\Phi}{dt} \quad [V] = -NA \frac{dB}{dt} = \frac{377}{2} \times 10^{-2} \pi \sin 377t(3a_y + 4a_z) \text{Tesla}
\]

\[= 1.88 \sin 377t(3a_y + 4a_z)T, \quad N=1\]

\[|\text{e.m.f}| = 1.88V\]

Q2:

- \( \mathbf{E} = 0.02 \sin [0.1927 \times (3\times10^8 t - z)] \mathbf{a}_x \)

\[
J_d = \frac{\partial}{\partial t} \mathbf{D} = \varepsilon_0 \frac{\partial}{\partial t} \mathbf{E} = \frac{10^{-9}}{36\pi} \times 0.02 \times 0.1927 \times 3 \times 10^8 \cos[0.1927(3 \times 10^8 t - z)]a_x
\]

\[= 1.008 \times 10^{-4} \cos[-] a_x\]

\[|J_d| = 0.1mA\]

- For good conductor

Q8:

\( \mathbf{E} = 10 \sin (\omega t - \beta y) \mathbf{a}_y, \, \text{V/m} \)
\( \mathbf{D} = \varepsilon_0 \mathbf{E}, \, \varepsilon_0 = 8.854 \times 10^{-12} \, \text{F/m} \)
\( \mathbf{D} = 10\varepsilon_0 \sin (\omega t - \beta y) \mathbf{a}_y, \, \text{C/m}^2 \)

Second Maxwell’s equation is: \( \nabla \times \mathbf{E} = -\mathbf{B} \)

As \( E_y = 10 \sin (\omega t - \beta z) \text{ V/m} \)

Now, \( \nabla \times \mathbf{E} \) becomes

\[= 10 \beta \cos (\omega t - \beta z) \mathbf{a}_x \]
That is, \( \nabla X E = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ O & E_x & O \end{vmatrix} \)

Or