MATH. (3)
LECTURE
NO. (9)
TRIPLE INTEGRALS
Triple Integrals:

\[ \iiint_V f(x, y, z) \, dv \]

Important properties of the Triple integral:

\[ \iiint_V a \cdot f(x, y, z) \, dv = a \iiint_V f(x, y, z) \, dv, \]

\[ \iiint_V (f(x, y, z) \pm g(x, y, z)) \, dv = \iiint_V f(x, y, z) \, dv \pm \iiint_V g(x, y, z) \, dv. \]

\[ \iiint_V f(x, y, z) \, dv = \iiint_{V_1} f(x, y, z) \, dv + \iiint_{V_2} f(x, y, z) \, dv \]
Evaluating the triple integral in rectangular coordinates

\[ \iiint_V f(x, y, z) \, dv \]

\[ \iiint_V f(x, y, z) \, dv = \int_{b \varphi_1(x)}^{b \varphi_2(x)} \int_{\varphi_1(x)}^{\psi_1(x,y)} \int_{\varphi_2(x)}^{\psi_2(x,y)} f(x, y, z) \, dz \, dA \]
Example

Evaluate the iterated integral \[ \iiint_V (x + y) \, dv \]
over the region \( V \) bounded by the planes:
\[ x = 0, \quad y = 0, \quad z = 0, \quad x + y + z = 1. \]

Solution:
\[
\iiint_V (x + y) \, dv = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x + y) \, dz \, dy \, dx
\]
\[
= \int_0^1 \int_0^{1-x} (x + y)(1 - x - y) \, dy \, dx
\]
Remark:

If \( f(x, y, z) = 1 \), then the volume of the solid \( V \) is

\[
V = \iiint_V dv
\]

If \( \rho(x, y, z) \) is density, then the mass of the solid \( V \)

\[
m = \iiint_V \rho(x, y, z) dv
\]

The coordinates of the center of the solid \( V \) are given by:

\[
x_c = \frac{1}{m} \iiint_V x \rho(x, y, z) dv, \quad y_c = \frac{1}{m} \iiint_V y \rho(x, y, z) dv,
\]

\[
z_c = \frac{1}{m} \iiint_V z \rho(x, y, z) dv.
\]
Example

Find the volume and the coordinates of the center of gravity of the region bounded by the parabolic cylinder:

\[ z = 4 - x^2 \] and \( x = 0 \), \( y = 0 \), \( y = 6 \), \( z = 0 \).

Assuming the density to be constant \( k \).

**Solution:**

\[
V = \iiint_{D} dz \, dy \, dx = \int_{x=0}^{x=2} \int_{y=0}^{y=6} \int_{z=0}^{z=4-x^2} dz \, dy \, dx
\]

\[
= \int_{x=0}^{x=2} \int_{y=0}^{y=6} (4-x^2) \, dy \, dx
\]

\[
= 6 \int_{x=0}^{x=2} (4-x^2) \, dx = 6 \left( 4x - \frac{1}{3}x^3 \right)_{0}^{2} = 32
\]
The mass of solid is:

\[
m = \iiint_{V} \rho(x, y, z) \, dx \, dy \, dz = \iiint_{V} k \, dx \, dy \, dz
\]

\[
= k \iiint_{V} \, dx \, dy \, dz = k \, V = 32k
\]

The coordinates of the centre of gravity:

\[
x_c = \frac{1}{m} \iiint_{V} \rho x \, dz \, dy \, dx = \frac{k}{32k} \iiint_{V} x \, dz \, dy \, dx
\]

\[
= \frac{1}{32} \iiint_{0}^{2} \iiint_{0}^{6} \int_{0}^{\sqrt{4-x^2}} x \, dz \, dy \, dx = \frac{4}{3}
\]
\[ y_c = \frac{1}{m} \iiint_V \rho y \, dz \, dy \, dx = \frac{k}{32k} \iiint_V y \, dz \, dy \, dx \]

\[ = \frac{1}{32} \int_0^2 \int_0^6 \int_0^{4-x^2} y \, dz \, dy \, dx = 3 \]

\[ z_c = \frac{1}{m} \iiint_V \rho z \, dz \, dy \, dx = \frac{k}{32k} \iiint_V z \, dz \, dy \, dx \]

\[ = \frac{1}{32} \int_0^2 \int_0^6 \int_0^{4-x^2} z \, dz \, dy \, dx = \frac{8}{5} \]

Then \((x_c, y_c, z_c)\) the center of gravity is \(\left(\frac{4}{3}, 3, \frac{8}{5}\right)\)
THE END