The first order differential equations may appear in one of the following forms:

\[ \frac{dy}{dx} = f(x, y) \]

\[ F\left(x, y, \frac{dy}{dx}\right) = 0 \]

\[ M(x, y)dx + N(x, y)dy = 0 \]
Methods of Solution of First Order ODE

(1) Separable Differential Equations:

\[ \frac{dy}{dx} = f(x, y) \implies \frac{dy}{dx} = \frac{g(x)}{h(y)} \implies \int h(y) \, dy = \int g(x) \, dx + c \]

\[ M(x, y) \, dx + N(x, y) \, dy = 0 \implies a(x)b(y) \, dx + c(x)d(y) \, dy = 0 \]
A separable differential equation can be expressed as the product of a function of $x$ and a function of $y$.

$$\frac{dy}{dx} = g(x) \cdot h(y) \quad h(y) \neq 0$$

Example:

$$\frac{dy}{dx} = 2xy^2$$

Multiply both sides by $dx$ and divide both sides by $y^2$ to separate the variables. (Assume $y^2$ is never zero.)

$$\frac{dy}{y^2} = 2x \; dx$$

$$y^{-2} \; dy = 2x \; dx$$
Separable Differential Equations

A separable differential equation can be expressed as the product of a function of $x$ and a function of $y$.

\[
\frac{dy}{dx} = g(x) \cdot h(y) \quad h(y) \neq 0
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Example:

\[
\frac{dy}{dx} = 2xy^2
\]

\[
\frac{dy}{y^2} = 2x \, dx
\]

\[
y^{-2} \, dy = 2x \, dx
\]

\[
\int y^{-2} \, dy = \int 2x \, dx
\]

\[
-y^{-1} + C_1 = x^2 + C_2
\]

\[
-\frac{1}{y} = x^2 + C
\]

\[
-\frac{1}{x^2 + C} = y
\]

\[
y = -\frac{1}{x^2 + C_6}
\]
Initial conditions

• In many physical problems we need to find the particular solution that satisfies a condition of the form $y(x_0)=y_0$. This is called an initial condition, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an initial-value problem.
Example:

\[
\frac{dy}{dx} = 2x(1 + y^2)e^{x^2}
\]

[Separable differential equation]

\[
\frac{1}{1 + y^2} dy = 2x e^{x^2} dx
\]

\[
\int \frac{1}{1 + y^2} dy = \int 2x e^{x^2} dx
\]

\[
\int \frac{1}{1 + y^2} dy = \int e^u du
\]

\[
\tan^{-1} y + C_1 = e^u + C_2
\]

\[
\tan^{-1} y + C_1 = e^{x^2} + C_2
\]

\[
\tan^{-1} y = e^{x^2} + C
\]

[Combined constants of integration]
Example (cont.):

\[ \frac{dy}{dx} = 2x(1 + y^2)e^{x^2} \]

\[ \tan^{-1} y = e^{x^2} + C \rightarrow \]

We now have \( y \) as an implicit function of \( x \).

\[ \tan\left(\tan^{-1} y\right) = \tan\left(e^{x^2} + C\right) \]

\[ y = \tan\left(e^{x^2} + C\right) \]

We can find \( y \) as an explicit function of \( x \) by taking the tangent of both sides.
Example

Solve the IVP \( \frac{dy}{dx} = e^{x+y-1}, y(0) = 1 \)

Solution

\[
\frac{dy}{dx} = e^{x+y-1} \quad \rightarrow \quad \frac{dy}{dx} = e^x e^{y-1}
\]

\[
\int e^x \, dx = \int e^{1-y} \, dy + c \quad \rightarrow \quad e^x = -e^{1-y} + c
\]

\[
x = 0 \Rightarrow y = 1 \quad \rightarrow \quad 1 = -1 + c \Rightarrow c = 2
\]

\[
e^x + e^{1-y} - 2 = 0 \quad \text{math(3)}
\]
Example

Solve the DE \( \left( x^2 y + 2x^2 \right) \frac{dy}{dx} = xy + 2y + x + 2 \)

Solution

\( \left( x^2 y + 2x^2 \right) \frac{dy}{dx} = xy + 2y + x + 2 \)

\( x^2(y + 2)dy = y(x + 2) + (x + 2)dx \)

\( x^2(y + 2)dy = (x + 2)(y + 1)dx \)

\( \frac{y + 2}{y + 1}dy = \frac{x + 2}{x^2}dx \quad \int 1 + \frac{1}{(y + 1)}dy = \int \frac{1}{x} + \frac{2}{x^2}dx + c \)

\( y + \ln(y + 1) = \ln x - \frac{2}{x} + c \)
(2) DE’s Reducible to Separable

This type takes the form:

\[
\frac{dy}{dx} = f(ax + by + c)
\]

Put \( u = ax + by + c \)

this transform the DE into separable one.
Example

Solve the DE \[ \frac{dy}{dx} = (2x + y + 7)^2 \]

Solution

let \( u = 2x + y + 7 \)

differentiate w.r.t. \( x \), we get:

\[ \frac{du}{dx} = 2 + \frac{dy}{dx} \]

\[ \frac{du}{dx} - 2 = u^2 \]

\[ \frac{du}{dx} = 2 + u^2 \]

\[
\int \frac{du}{2 + u^2} = \int dx + c
\]

\[
\frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{u}{\sqrt{2}} \right) = x + c
\]

\[
\frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{2x + y + 7}{\sqrt{2}} \right) = x + c
\]
Example

Solve the DE \( \frac{dy}{dx} = \sin(x + y + 1) \)

**Solution**

*let* \( u = x + y + 1 \)

differentiate w.r.t. \( x \), we get:

\[ \frac{du}{dx} = 1 + \frac{dy}{dx} \]

\[ \frac{du}{dx} - 1 = \sin u \]

\[ \frac{du}{dx} = 1 + \sin u \]

\[ \int \frac{du}{1 + \sin u} = \int dx + c \]

\[ \int \frac{1 - \sin u}{1 - \sin^2 u} du = \int dx + c \]

\[ \int \sec^2 u - (\cos u)^{-2} \sin u \, du = \int dx + c \]

\[ \tan u + (\cos u)^{-1} = x + c \]
(3) Homogeneous DE’s

This type takes the form:

\[ \frac{dy}{dx} = f \left( \frac{y}{x} \right) \]

The right hand side is a homogeneous function of degree zero.

\[ u = \frac{y}{x} \]

This transform the DE into separable one.
Example

Solve the DE: \[ x \frac{dy}{dx} = y + x e^x \]

Solution

let \( \frac{y}{x} = u \) \( \Rightarrow y = ux \)

differentiate w.r.t. \( x \), we get:

\[ \frac{dy}{dx} = u + x \frac{du}{dx} \]

\[ u + x \frac{du}{dx} = u + e^u \]

\[ x \frac{du}{dx} = e^u \]

\[ \int e^{-u} du = \int \frac{1}{x} dx + c \]

\[ -e^{-u} = \ln x + c \]
Example

Solve the DE:

\[
\frac{dy}{dx} = \frac{x + y}{x - y}
\]

\[
\frac{dy}{dx} = \frac{1 + \frac{y}{x}}{1 - \frac{y}{x}}
\]

Solution

Let \( \frac{y}{x} = u \Rightarrow y = ux \)

differentiate w.r.t. x, we get:

\[
\frac{dy}{dx} = u + x \frac{du}{dx}
\]

\[
u + x \frac{du}{dx} = \frac{1 + u}{1 - u}
\]

\[
x \frac{du}{dx} = \frac{1 + u - u + u^2}{1 - u}
\]

\[
x \frac{du}{dx} = \frac{1 + u}{1 - u}
\]

\[
\int \frac{1-u}{1+u^2} \, du = \int \frac{1}{x} \, dx + c
\]

\[
tan^{-1} u - \frac{1}{2} \ln(1 + u^2) = \ln x + c
\]
The slope $m$ of a curve is 0 where the curve crosses the $y$-axis, and $\frac{dy}{dx} = \sqrt{1 + y^2}$. Find $m$ as a function of $x$.

**SOLUTION**

$$\frac{dy}{\sqrt{1+y^2}} = dx$$

Integrate both sides

$$\int \frac{dy}{\sqrt{1+y^2}} = \int dx + c$$

$\sinh^{-1}(y) = x + c$

$y=0$ at $x=0$ then $c=0$ and $y=\sinh(x)$
THE END